Lecture 14: Reading off a general solution

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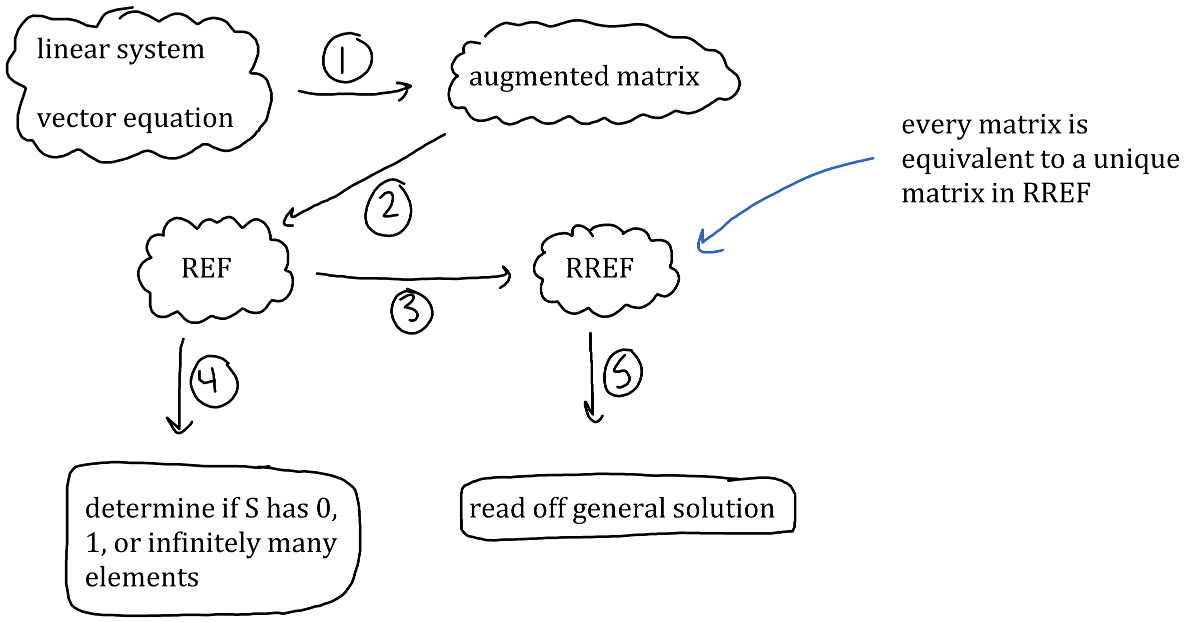
Mindmap of what we're currently doing:



(1) Copy coefficient and solution vector
(2) and **(3)** Gaussian Elimination (handout from last lecture)
(4) and **(5)** Reading off a general solution

11.6 Reading off a general solution

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 & 2 \end{array}\right] \end{array}$$ → in REF but not in RREF

$$\begin{array}{c} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 2 \end{array}\right] \\ \begin{array}{cccc} N & P & P & N \end{array} \end{array}$$ → in RREF

N="non-pivot": column without a leading 1
    ie. a free parameter
P="pivot": column with a leading 1
    ie. no choice

Choose free parameters:

$x_1 = s \qquad x_2 + 3t = 0 \quad \Rightarrow \quad x_2 = -3t$

$x_4 = t \qquad x_3 = 2 \quad \Rightarrow \quad x_3 = 2$

$$\Rightarrow S = \left\{ \begin{pmatrix} s \\ -3t \\ 2 \\ t \end{pmatrix} \middle| \; s, t \in \mathbb{R} \right\}$$

Another example

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & a & 0 & b & d \\ 0 & 0 & 1 & e & f \\ 0 & 0 & 0 & 0 & g \end{array}\right] \\ \begin{array}{cccc} & s & & t \end{array} \end{array}$$

Here there are two different solutions depending on the value of g:

a) $g \neq 0 \quad \Rightarrow \quad S = \{\square\} = \emptyset$

b) $g = 0 \quad \Rightarrow \quad x_2 = s \qquad x_1 + a * s + b * t = d$

$\qquad\qquad\qquad x_4 = t \qquad x_3 + e * t = f$

b) $g = 0 \Rightarrow$ $x_2 = s$, $x_4 = t$ $\quad$ $x_1 + a * s + b * t = d$, $x_3 + e * t = f$

$x_1 = d - as - bt$
$x_3 = f - et$

$$S = \left\{ \begin{pmatrix} d-as-bt \\ s \\ f-et \\ t \end{pmatrix} \middle| s, t \in \mathbb{R} \right\}$$

<u>Another example</u>

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right] \Rightarrow \begin{array}{l} x_1 = a \\ x_2 = b \\ x_3 = c \end{array} \Rightarrow S = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\}$$

Rules

(1) Row of type "$0, \dots, 0 \mid \underset{}{\overset{\neq 0}{*}}$"? $\quad$ "degenerate inhomogeneous equation"

<u>Yes</u>: $S = \emptyset$

<u>No</u>: Continue

(2) Each column of the coefficient matrix has a leading 1?

<u>Yes</u>: S has 1 element

<u>No</u>: S has infinitely many elements

(1) and (2) work with matrixes in REF. For (3) onwards, RREF is required:

(3) If S has 1 element, solution is the vector in the augmented column. $\quad$ ignore zero-rows

(4) If S has infinitely many elements, the variables corresponding to columns without leading 1s become free parameters (generally s and t). The remaining variable can now be expressed in terms of the free parameters.

Examples

Find a linear system with m=4 equations and n=3 variables that has:

		rank (A)	rank (A|b)
0 solutions (inconsistent)	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 4 \end{array}\right]$	3	4
1 solution	$\left[\begin{array}{cccc\|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$	3	3
infinitely many solutions	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$	2	2

Find a homogenous linear system with 4 equations and 3 variables that has:

0 solutions $\quad$ —— DNE

1 solution

0 solutions —— DNE

1 solution

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad 3 \qquad 3$$

infinitely many solutions

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad 2 \qquad 2$$

$\forall$ rank(A) $\underset{\substack{\downarrow \\ \text{always!}}}{\leq}$ rank(A|b)

Chapter 12

Definition
The rank of a matrix A, denoted by rank(A), is the number of leading 1s (pivots) in any REF of A.

Theorem
Let $[A|b]$ be an augmented matrix. Then:

a) 0 solutions (inconsistent) <=> rank(A) < rank(A|b)

b) 1 solution <=> rank(A)=rank(A|b) and rank(A)= # columns of A

c) infinitely many solutions <=> rank(A)=rank(A|b) and rank(A) < # columns of A

See textbook for Chapter 13: Applications